MA 335, Spring 2012

Assignment 8.

This homework is due *Thursday* March 22.

There are total 37 points in this assignment. 33 points is considered 100%. If you go over 33 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

(1) [4pt] (6.2.2) The *Mangoldt function* Λ is defined by
$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \text{ where } p \text{ is prime and } k \geq 1; \\ 0, & \text{othewise.} \end{cases}$$
Prove that $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = -\sum_{d|n} \mu(d) \log d$.
(*Hint:* Show $\sum_{d|n} \Lambda(d) = \log n$ and then apply the Möbius inversion formula.)

(2) [4pt] (6.2.3) Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that
$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r)).$$
(*Hint:* Left hand side is $(\mu f) * \mathbf{1}$ so it is multiplicative. Check that the equality holds for $n = p^k$ and argue that it suffices.)

(3) (6.2.4abd) For the integer $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, use item problem 2 to establish the following:
- (a) [1pt] $\sum_{d|n} \mu(d)\tau(d) = (-1)^r$,
- (b) [1pt] $\sum_{d|n} \mu(d)\sigma(d) = (-1)^r p_1 p_2 \cdots p_r$,
- (c) [1pt] $\sum_{d|n} \mu(d) d = (1 - p_1)(1 - p_2) \cdots (1 - p_k)$.

(4) [2pt] (7.2.1) Calculate $\varphi(1001)$, $\varphi(5040)$, $\varphi(36000)$.

(5) [3pt] (7.2.8) Prove that if the integer n has at least r distinct odd prime factors, then $2^r \mid \varphi(n)$.

(6) [3pt] (7.2.10) If every prime that divides n also divides m, establish $\varphi(mn) = n\varphi(m)$; in particular, $\varphi(n^2) = n\varphi(n)$ for every positive integer n.

(7) [3pt] (7.2.13) Prove that if $d \mid n$, then $\varphi(d) \mid \varphi(n)$. (*Hint:* Work with the prime factorizations of d and n.)

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(8) (7.2.14)

(a) [3pt] Prove that for the positive integers m, n, where $d = \gcd(m, n)$,

$$\varphi(m)\varphi(n) = \varphi(mn)\frac{\varphi(d)}{d}.$$

(b) [3pt] Prove that for the positive integers m, n,

$$\varphi(m)\varphi(n) = \varphi(\gcd(m, n))\varphi(\operatorname{lcm}(m, n)).$$

(9) [3pt] (7.3.2) Use Euler's theorem to prove that for any integer $n \geq 0$,

$$51 \mid 10^{32n+9} - 7.$$

(10) [3pt] (7.3.9) Use Euler's theorem to evaluate remainder of 2^{100000} mod 77.

(11) [3pt] (7.3.10) For any integer a, show that a and a^{4n+1} have the same last digit.